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Sequential Bayesian Persuasion*

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Abstract

I study a Bayesian persuasion model in which multiple senders sequentially persuade one receiver, after observing signal structures of prior senders and their realizations. I develop a geometric method, *recursive concavification*, to characterize the Subgame Perfect Equilibrium paths. I show that if there are two senders who have constant-sum payoffs, the truth-telling signal structure is always supported in equilibrium. I prove the existence of the *silent equilibrium*, where at most one sender provides nontrivial information. I also provide a sufficient condition under which it is without loss of generality to focus on silent equilibria.

Keywords: Bayesian Persuasion, Multiple Senders, Subgame Perfect Equilibrium, Communication.

JEL Classification Codes: D82, D83

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1 Introduction

When one agent makes an attempt to persuade a decision maker, he might have a concern about how other agents would follow up on his information and provide additional information to the decision maker. In this situation, the information revelation is largely shaped by strategic interaction between agents with potential conflicts of interest.

To understand the importance of strategic interaction, let us review the motivating example in Kamenica and Gentzkow (2011). There is a defendant who is either innocent or guilty. A judge wants to convict the guilty and acquit the innocent, while a prosecutor wants to persuade the judge to convict the defendant. The prosecutor can choose a process for gathering evidence about the defendant and present that evidence to the judge. This process is what I refer to as a signal structure. Once the prosecutor chooses the signal structure, he cannot hide or alter the signal realization. However, he can choose a biased signal structure, in which the *ex ante* probability of sending a *convict* signal is higher than the true probability that the defendant is guilty. The judge realizes the extent to which the signal structure is biased. But, a *convict* signal still implies that the person is more likely to be guilty, so the judge follows the suggestion from the prosecutor. If, under the prior, the judge would acquit, the prosecutor benefits from manipulating the signal structure.

Suppose now there is a defense attorney who desires acquittal. After observing the signal structure chosen by the prosecutor and the signal realization, the attorney also chooses a signal structure. The picture dramatically changes. If the prosecutor continues to use a biased signal structure, the judge would be relatively uncertain, and the attorney's information becomes more valuable. That means the judge would be inclined to follow the suggestion from the attorney, even though the attorney may use a signal structure largely biased in his interest — that is, a signal structure that sends an *acquit* signal with higher probability than the true probability that the defendant is innocent. With this concern, it is uniquely optimal for the prosecutor to fully reveal the information in the first place.

In this paper, I extend the canonical model of Bayesian persuasion (Kamenica and Gentzkow, 2011) to the case of multiple senders, where the senders move sequentially in a fixed order. A typical Bayesian persuasion analysis involves the receiver breaking ties in favor of a certain sender. In Section 3, I will show that it is *with loss* to impose a tie-breaking rule, in the sense that it would preclude certain equilibrium of interest. However, I do not make such an assumption; instead, I will characterize the full set of Subgame Perfect Equilibrium (SPE) paths in Section 6.

With this in mind, I develop a novel method, called *recursive concavification*, to characterize the set of SPE paths. The key of recursive concavification is to decompose the problem into single-sender problems and then solve them one-by-one in a backward manner. Note that the last sender always faces the same problem as in a single-sender game. The previous play only influences his decision, in that his belief is updated from prior information. This amounts to a new "prior" for the single-sender game. To find the equilibrium signal structure, I obtain the set of the sender's continuation payoffs ("value correspondence") derived from the receiver's best responses to her beliefs, and impose concave closure on the minimum of this correspondence on the belief space. Then, in the spirit of Harris (1985), I solve for the SPE paths of the continuation game beginning at the penultimate period. Based on the characterization of SPE paths of the last two periods, I obtain the second to last sender's continuation payoffs in the subsequent continuation games. Since the continuation payoffs are the only requirement needed to apply the concavification method, this is equivalent to solving this sender's problem as a single sender problem. By repeating these steps, I can solve for the SPE paths of the whole game.

Next, I use this characterization to address two questions. In Section 7, I prove the existence of a fully revealing equilibrium when there are two senders with constant-sum utilities. Note, there may be more than two senders. What is required is that there are two senders with conflicting interest. The result coincides with the insight in the aforementioned example of the prosecutor and attorney. The reason is that, even when there are more than two senders, the incentive of those with constant-sum utilities is neither effected by the other senders' activity nor the order of moves.

In Section 8, I examine a particular kind of equilibria, called a *silent equilibrium*. In a silent equilibrium, at most one sender reveals information on the equilibrium path. I, first, prove the existence of such an equilibrium. The basic idea is to find an optimal signal structure for each sender such that it already contains information the subsequent senders would like to release, were the information not revealed. Hence, the incentives of subsequent senders to communicate disappear *on the equilibrium path*. Second, I give a sufficient (but not a necessary) condition under which it is without loss of generality to focus on silent equilibria. Many economically interesting situations satisfy this condition, such as the motivating example in Board and Lu (2018),[1] the above prosecutor-attorney example, and a conflicting interest game in Section 7.

[1]Board and Lu (2018) studies a search model.

The remainder of the paper is organized as follows. Section 2 summarizes the existing literature that has relation to the current paper. Section 3 uses a simple example to illustrate the main idea. Section 4 lays out the basic model. Sections 5 and 6 characterize the set of SPE paths in the cases of single sender and multiple senders, respectively. Section 7 shows how competition among two senders with constant-sum utilities yields a fully revealing equilibrium. Section 8 proves the existence of a silent equilibrium and provides a condition for outcome equivalence between silent equilibria and SPE. All proofs are relegated to the Appendix.

2 Literature Review

There has been a growing literature on Bayesian persuasion with multiple senders (Au and Kawai, 2020; Board and Lu, 2018; Gentzkow and Kamenica, 2017a,b; Li and Norman, 2020; Ravindran and Cui, 2020). Gentzkow and Kamenica (2017a,b) focus on the case where multiple senders move simultaneously and explore conditions under which competition among senders improves information transparency. Board and Lu (2018) study Bayesian persuasion in a search model, and provide conditions for the existence of a fully revealing equilibrium. Au and Kawai (2020) analyze a model where multiple senders compete in disclosing positive information about the quality of their products. Ravindran and Cui (2020) study a simultaneous-move persuasion game in which senders with zero-sum payoffs construct independent signal structures. They focus on the condition for *uniqueness* of fully revealing equilibrium.

In simultaneous and independent work, Li and Norman (2020) also study the game of multiple senders who move in a sequence. They assume that the receiver plays pure strategies and breaks ties in favor of the last sender. Under this assumption, they are able to simplify the problem into a linear programming task and reach strong results, such as the outcome equivalence between SPE and silent equilibria and the essential uniqueness of the silent equilibrium. However, this assumption could be restrictive. In Section 3, I will provide an example that illustrates a substantively interesting equilibrium that is not captured by the analysis in Li and Norman (2020). In Appendix D.1, I will present a counterexample in which an SPE outcome cannot be achieved by any silent equilibria.

Methodologically, the method of recursive concavification draws on results from the literature of SPE of infinite games (Harris, 1985; Hellwig, Leininger, Reny, and

Robson, 1990). Harris (1985) points out that the necessary and sufficient condition for a strategy of a player moving at a certain period to be supported in an equilibrium of an infinite game can be indicated by a lower bound for the player's continuation payoff. This lower bound is the highest payoff the player can guarantee himself given that subsequent players coordinate in the worst SPE for him in the following continuation games. Surprisingly, this lower bound has a geometric representation in Bayesian persuasion games, that is, the concave closure over the minimal continuation payoff function. If a sender designs a signal structure, provided the subsequent players coordinate in SPE, that would give him an expected payoff not lower than the lower bound, this signal structure is supported on an SPE path.

This paper also broadly relates to communication models in extensive form games and dynamic programming. Aumann and Hart (2003) and Forges and Koessler (2008) provide geometric characterizations of equilibrium outcomes in long cheap talk and long disclosure games, respectively. They show how multistage exchanges of verifiable or unverifiable messages expand the set of equilibrium outcomes. Krishna and Morgan (2001) study a model in which a decision maker sequentially consults two experts who are privately informed and can send cheap talk messages. When the two experts are biased in opposite directions, they will reveal more information than if there were a single sender. Ely (2017) analyzes a dynamic persuasion problem in which a principal releases information about a stochastic process to an agent who makes decisions in each period. In approaching the optimal persuasion scheme, the author adopts a similar method to this paper, by repeatedly concavifying the sender's continuation payoff function. Pavan and Calzolari (2009) investigate a contracting problem between a sequence of principals and one common agent. They discuss to what extent the *menu* theorems in common agency problem can be applied to the extensive form game.

3 Example

In this section, I am going to characterize the set of SPE paths of a specific example. In particular, I find that to support some equilibrium, the receiver may need to randomize between different actions on the equilibrium path. This is inconsistent with any tie breaking rule for receiver that favors a certain sender.

Consider an entry game with one entrant, E, and two incumbents, I_1 and I_2. Whether E wants to enter depends on the state of the market, which could be *good*

		Action			
		In	A_1	A_2	*Out*
State	*Good*	4,4,4	13,4,3	4,13,3	13,13,0
	Bad	0,0,-2	1,0,-1	0,1,-1	1,1,0

Table 1: Payoff Table for the Incumbents and the Entrant

or *bad*. There is a common prior $\mu_0(bad) = \frac{1}{3}$. E can stay out of the market, go in independently (In), ally with I_1 (A_1), or ally with I_2 (A_2).

Table 1 presents players' payoffs. Each table entry contains the payoffs for I_1, I_2, and E, respectively. Notice the following features of the payoff structure. Each incumbent prefers the entrant allying with himself or staying out, to her allying with his opponent or operating independently. While their payoffs are negatively affected by a bad state.

Let μ_t $(t = 1, 2)$ be the posterior probability that the state is bad after period t. The entrant's best responses depend on the second period belief μ_2. The entrant would like to choose In when $\mu_2 \leq \frac{1}{2}$, A_1 or A_2 when $\mu_2 \in [\frac{1}{2}, \frac{3}{4}]$, and Out when $\mu_2 \geq \frac{3}{4}$.

The state is unknown to all firms. But the incumbents can produce signal structures that generate signals about the state. A signal structure π maps states onto distributions over a signal space $\{G, B\}$, i.e., $\pi : \{good, bad\} \to \Delta(\{G, B\})$. The firms move sequentially. I_1 chooses π_1 that sends a signal s_1. I_2 observes (π_1, s_1), and then chooses π_2 that sends s_2. Finally, E observes (π_1, s_1, π_2, s_2), and then takes an action from $\{In, A_1, A_2, Out\}$.

Given any μ_2, E's best responses induce a set of continuation payoffs for I_2. Figure 1 shows the set of I_2's payoffs given E's best response as a function of μ_2. When $\mu_2 \in [\frac{1}{2}, \frac{3}{4}]$, E has two optimal choices, A_1 and A_2. Since she can randomize between them, I_2's feasible continuation payoffs are between the expected payoffs deriving from A_1 and A_2. Differently from this paper, Kamenica and Gentzkow (2011) and Li and Norman (2020) restrict E to take A_2 when she is indifferent, which leads to the maximal payoff function on the belief space.

At the start of period 2, the game can be seen as one that involves a single sender, I_2, with a common prior μ_1. I first discuss the lower bound on I_2's equilibrium payoffs given each μ_1. When $\mu_1 \in [0, \frac{3}{4}]$, I_2 should get an expected payoff at least 4.

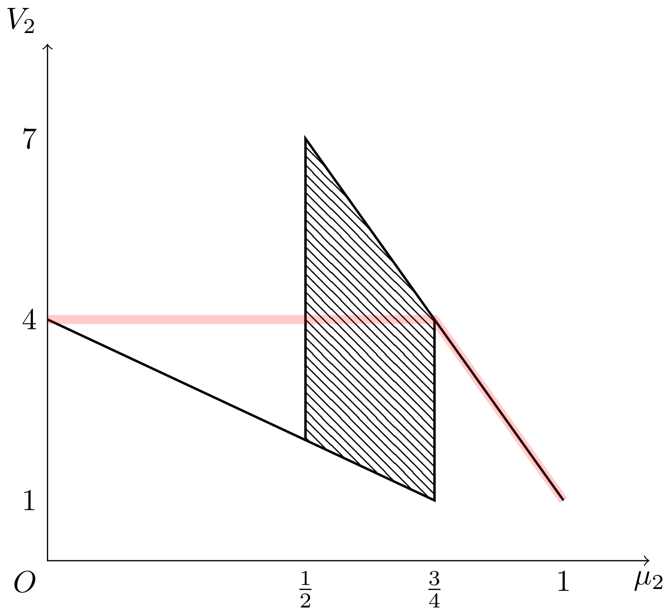

Figure 1: I_2's Continuation Payoffs

Otherwise, I_2 can deviate to another signal structure that induces posterior beliefs $\mu_2 = 0$ and $\mu_2 = \frac{3}{4} + \epsilon$, and will receive an expected payoff which converges to 4 when ϵ goes to 0. When $\mu_1 \in (\frac{3}{4}, 1]$, his equilibrium payoff is uniquely determined, since it is in his best interest to babble and let E stay out. This lower bound is indicated by the red curve in Figure 1. A general result in Section 6 will show a sufficient condition to characterize equilibrium strategies. That is, any signal structure, followed by the entrant's best responses, that gives I_2 an expected payoff higher than the lower bound is an equilibrium strategy.

Next, by using the lower bound, I can geometrically characterize the SPE paths of any continuation games beginning at period 2. Notice three typical types of SPE paths in different continuation games. First, consider the case where $\mu_1 = 0$. In that case, after the first period, the state is known to be *good*. So, I_2's signal has no effect and E will enter independent of the signal. Thus, I_1's payoff is 4. Second, consider the case where $\mu_1 \in [\frac{1}{2}, \frac{3}{4}]$. In that case, I_2 babbles, which results in $\mu_2 = \mu_1$. There is an SPE of this continuation game where I_2 babbles and, with probability at least $\frac{9}{5} - \frac{12}{5}\mu_1$, E allies with I_2. Under that strategy profile, I_2 has an expected payoff above 4; this meets the lower bound for SPE paths. Third, consider the case where $\mu_1 \in (\frac{3}{4}, 1]$. In that case, I_2 babbles and E stays out. So, I_1's expected payoff is $13 - 12\mu_1$. I_1's equilibrium payoffs on these three types of SPE paths are illustrated by the black point, the shaded area, and the solid line in Figure 2.

Of course, the set of the SPE paths is more abundant than I have described.

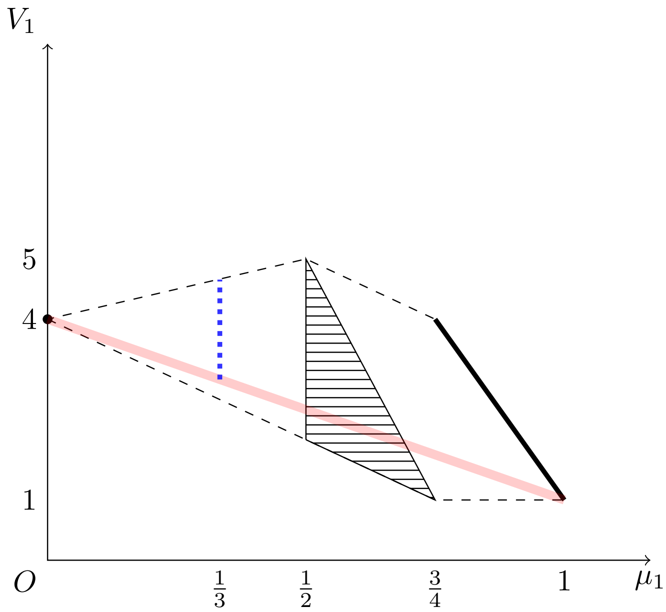

Figure 2: I_1's Continuation Payoffs

Yet, in terms of I_1's continuation payoffs, it suffices to create the convex hull of the above three areas, as indicated by the dashed lines in Figure 2. That is because I_2 can create a signal structure that randomizes between different SPE paths, which itself forms an SPE path. Therefore, I_1's payoff is the convex combinations of those on the original SPE paths. It can be checked that I_1's equilibrium payoff does not go beyond this convex hull.

After obtaining I_1's value correspondence, I can solve for his equilibrium strategies. It turns out that I_1's equilibrium payoffs should be bounded below by the payoffs deriving from the fully revealing strategy. This lower bound is indicated by the red curve in Figure 2. Again, this condition is not only necessary, but also sufficient. That means the range of I_1's equilibrium payoffs is $[3, \frac{14}{3}]$, as shown by the blue dotted line in Figure 2. The SPE path that achieves his highest payoff is specified as below. I_1 uses a signal structure π_1 such that

$$\pi_1(G|good) = \frac{1}{2} \qquad \pi_1(B|good) = \frac{1}{2}$$
$$\pi_1(G|bad) = 0 \qquad \pi_1(B|bad) = 1$$

By Bayes rule, if the signal realized is G, that means the true state is *good*; while if the signal realized is B, the posterior is that $\mu_1 = \frac{1}{2}$. On the equilibrium path, I_2 babbles regardless of which signal is sent, and E chooses In after G and $\frac{3}{5}A_1 + \frac{2}{5}A_2$ after B.

Note that I_1's favorite equilibrium outcome involves E breaking ties by taking a mixed action, not completely in favor of either incumbent. That is different from Li and Norman (2020), who require that E breaks ties in favor of I_2. Therefore, this paper provides a characterization of a broader set of equilibrium outcomes in a sequential Bayesian persuasion game.

4 Model

In this section, I lay out a sequential game with multiple senders. The states of the world are $\Omega = \{\omega_1, \ldots, \omega_N\}$, $N \in \mathbb{N}$. Players have a common prior, $\mu_0 \in \text{int}(\Delta(\Omega))$. There are T senders and 1 receiver who move in a sequence. Each sender has access to a set of costless signal structures Π. A signal structure $\pi : \Omega \to \Delta(S)$ maps each state ω into a probability distribution over the finite signal space S, where $|S| \geq N + T$. Each signal structure π induces from a prior belief μ a distribution over posterior beliefs $\tau(\cdot|\pi, \mu) \in \Delta(\Delta(\Omega))$, which is called an *information policy.* Since the signal space is finite, throughout this paper I only consider information policies with finite support, such that $|\text{supp}(\tau)| \leq |S|$. By Bayes rule, the expectation of an information policy equals the prior, that is,

$$\sum_{\mu' \in \text{supp}(\tau)} \tau(\mu') \cdot \mu' = \mu \tag{1}$$

Conversely, for any distribution of posterior beliefs with support no greater than $|S|$ satisfying Eq. (1), there is a signal structure the sender can choose to induce it (Aumann and Maschler, 1995; Kamenica and Gentzkow, 2011). In other words, such a distribution of beliefs constitutes an information policy.

The receiver takes an action a from a finite set A. Utility functions of all players depend on the state and the action. The senders' utility functions are denoted by $v^t(a, \omega), t = 1, \ldots, T$, and the receiver's utility function $u(a, \omega)$. All utility functions extend naturally to expected utility functions of mixed actions.

The timing of the game is as follows:

Date 0 Nature picks a state ω according to μ_0. The true state ω is unknown to all players.

Date 1 Sender 1 chooses $\pi_1 \in \Pi$ which generates $s_1 \in S$.

Date 2 After observing (π_1, s_1), sender 2 chooses $\pi_2 \in \Pi$ which generates $s_2 \in S$.

. . .

Date T After observing $(\pi_1, s_1, \pi_2, s_2, \ldots, \pi_{T-1}, s_{T-1})$, sender T chooses $\pi_T \in \Pi$ which generates $s_T \in S$.

Date $T+1$ After observing $(\pi_1, s_1, \pi_2, s_2, \ldots, \pi_T, s_T)$, the receiver takes an action $a \in A$.

As the game unravels, more information is released and players' beliefs keep changing. Denote the updated belief after period t by μ_t. From period t onward, μ_{t-1} can be interpreted as the new "prior," and the signal structure π_t will further lead to $\tau_t(\cdot|\pi_t, \mu_{t-1})$ as a distribution over μ_t. A history up until the end of date t consists of a sequence of signal structures and their realizations, which is denoted by $h_t = (\pi_1, s_1, \ldots, \pi_t, s_t)$, $h_t \in H_t$. Let h_0 be the initial node. Conditional on any history h_t, there is a unique belief $\bar{\mu}(h_t)$ associated with the information set updated by Bayes rule.

The (mixed) strategy of sender t is a mapping $\sigma_t : H_{t-1} \to \Delta(\Pi)$ and the (mixed) strategy of the receiver $\rho : H_T \to \Delta(A)$. Denote the strategy set of sender t by Σ_t and that of the receiver Σ_R. Fixing the belief system $\bar{\mu}$, under a strategy profile (σ, ρ) the expected payoff for sender t conditional on a history h is represented by $\bar{v}^t(h; \sigma, \rho)$. The solution concept of this paper is given below.

Definition 1. *A Subgame Perfect Equilibrium (SPE) is a strategy profile* $(\sigma_1, \ldots, \sigma_T, \rho)$ *such that for each* $t = 1, \ldots, T$, $h_t \in H_t$, *and* $\sigma'_t \in \Sigma_t$, *it satisfies that:*

$$\mathbb{E}[v^t(a,\omega) \,|\, \rho, \sigma_{-t}, \sigma_t(h_t), \bar{\mu}(h_t)] \geq \mathbb{E}[v^t(a,\omega) \,|\, \rho, \sigma_{-t}, \sigma'_t, \bar{\mu}(h_t)]$$

and for each $\alpha \in \Delta(A)$ *and* $h_T \in H_T$,

$$\mathbb{E}[u(\rho(h_T), \omega) \,|\, \bar{\mu}(h_T)] \geq \mathbb{E}[u(\alpha, \omega) \,|\, \bar{\mu}(h_T)]$$

Following any history h_t, the remaining game takes the same form of the original game with a common prior $\bar{\mu}(h_t)$ and players including senders $t+1$, ..., T, and the receiver. Denote this continuation game by $G(h_t)$, and an SPE path of the game by $\bar{\gamma}(h_t)$. Finally, let $\Gamma(h_t)$ and $\bar{\Gamma}(h_t)$ represent the sets of SPE and SPE paths of the continuation game $G(h_t)$, respectively. In the next two sections, I will discuss how to characterize $\bar{\Gamma}(h_t)$, for any t and $h_t \in H_t$, recursively.

5 Single Sender

In this section, I have a review of the standard Bayesian persuasion problem (Kamenica and Gentzkow, 2011) and introduce a novel method to analyze the full set of SPE paths.

There is one sender who persuades one receiver. After collecting information from the sender, the receiver updates her belief to μ and has a set of best responses $r(\mu) \subseteq \Delta(A)$. Based on the receiver's best responses, the sender's continuation payoff correspondence ("value correspondence") is denoted by $V(\mu) = \{\mathbb{E}[v(\alpha, \omega) \,|\, \mu] \mid \alpha \in r(\mu)\}$. By the Maximum Theorem, V is nonempty valued and has closed graph (Appendix A.1). Let $\bar{V}$ and $\underline{V}$ represent the pointwise maximal and minimal values of V.

Next, I introduce the definition of the *concave closure* of a given function f, $\mathrm{cl}(f)$. There are two equivalent definitions according to Hiriart-Urruty and Lemaréchal (2012) pp.99, Proposition 2.5.1..The most well known definition describes it as the lowest concave function that dominates a certain function pointwise. However, I mainly use another definition, which states that the concave closure is the supremum of the convex combinations of function values.[2]

Definition 2. *If X is a convex compact measurable set, for any function $f : X \to \mathbb{R}$, its concave closure $\mathrm{cl}(f)$ is a function on X such that at any $x \in X$*

$$\mathrm{cl}(f)(x) = \sup_{\substack{q \in \Delta(X) \\ \mathbb{E}[q] = x}} \sum_{q' \in \mathrm{supp}(q)} q(q') \cdot f(q')$$

Under an SPE (σ, ρ), its equilibrium path consists of a signal structure π that yields an information policy τ, and a set of actions $\{\bar{\alpha}(s)\}_{s \in S}$ associated with each realized signal under π. Indeed, the concave closure of the minimal value $\mathrm{cl}(\underline{V})$ is the threshold for determining an equilibrium path.

Lemma 1. *$(\pi, \{\bar{\alpha}(s)\}_{s \in S})$, where π induces an information policy τ, is an SPE path if and only if*

1. *$\bar{v}(\pi, \bar{\alpha}) \geq \mathrm{cl}(\underline{V})(\mu_0)$.*

2. *For any $s \in S$, $\bar{\alpha}(s) \in r(\bar{\mu}(\pi, s))$.*

[2] The original definition calculates the convex combination of countably many elements. In this paper I assume a finite signal space, that means the cardinality of $\mathrm{supp}(q)$ is finite, too. However, by Carathéodory Theorem, it suffices to use $|\Omega|$ elements to approximate the supremem. So q can be viewed as an information policy whose support has $|\Omega|$ elements, which will not affect the definition of the concave closure.

The existence of an SPE is derived from the closedness of V (Appendix A.3). The *maximal* value function of V, $\bar{V}$, is well defined on $\Delta(\Omega)$ and upper semi-continuous (USC). Hence, the value on the concave closure, $\text{cl}(\bar{V})(\mu_0)$, is achievable by using some signal structure and is certainly no less than $\text{cl}(\underline{V})(\mu_0)$.

Lemma 2. *For any $\mu_0 \in \text{int}(\Delta\Omega)$, an SPE exists.*

The range of SPE payoffs for the sender is sandwiched by the concave closure of the maximal and minimal value functions (Appendix A.4). As has been discussed above, $\text{cl}(\bar{V})(\mu_0)$ and $\text{cl}(\underline{V})(\mu_0)$ are equilibrium payoffs. Then, the sender can construct a signal structure that randomizes across the two equilibrium signal structures underlining the equilibrium paths. Let the receiver respond in the same way as on the two equilibrium paths. Therefore, any payoff in between can be achieved in such a construction where the sender's payoff is not lower than the threshold, $\text{cl}(\underline{V})(\mu_0)$, and the receiver best responds.

Proposition 1. *The set of SPE payoffs to the sender is* $[\text{cl}(\bar{V})(\mu_0), \text{cl}(\underline{V})(\mu_0)]$.

6 Multiple Senders

In this section, I introduce more senders into discussion and show that the key results in the case of single sender carry over to the more complicated situation. I will characterize the set of SPE paths by recursively applying the geometric method.

Initially, define the set of the continuation payoffs for sender k conditional on an updated belief μ after period t as

$$V_t^k(\mu) = \{\bar{v}^k(h_t;\sigma,\rho) \,|\, \bar{\mu}(h_t) = \mu, (\sigma_{t+1},\dots,\sigma_T,\rho) \in \Gamma(h_t)\} \tag{2}$$

In the last period, the value correspondence for Sender T, V_T^T, is calculated in the same way as that in the single sender case, which only depends on the receiver's best responses. Therefore, I can use $\text{cl}(\underline{V}_T^T)$ as the criterion to solve for the set of equilibrium paths $\bar{\Gamma}(h_{T-1})$ for any $h_{T-1} \in H_{T-1}$. Once I obtain $\bar{\Gamma}(h_{T-1})$, the value correspondences of the penultimate period, $\{V_{T-1}^k\}_{k=1}^T$, are well defined. Similar to the single sender case, V_{T-1}^k has two properties: (1) It is non-empty valued. Because according to Lemma 2, an SPE exists in every continuation game $G(h_{T-1})$. (2) It has closed graph. Because the set of equilibrium paths is upper hemi-continuous.

Consider the persuasion problem of sender $T-1$. It is equivalent to a single sender game with a common prior $\bar{\mu}(h_{T-2})$ and a value correspondence V_{T-1}^{T-1}. Note that the value correspondence that summarizes his continuation payoffs in following continuation games is essential to determine his optimal strategies. Again, to reach the concave closure of the minimal value function $\text{cl}(\underline{V}_{T-1}^{T-1})$ is the necessary and sufficient condition for the characterization of $\bar{\Gamma}(h_{T-2})$. After obtaining $\bar{\Gamma}(h_{T-2})$, $\{V_{T-2}^k\}_{k=1}^T$ are well defined, and I can move on to characterize $\bar{\Gamma}(h_{T-3})$ with the similar method, so on and so forth. Finally, I can characterize the set of SPE paths of the original game, $\bar{\Gamma}(h_0)$.

Theorem 1. *The set of SPE paths $\bar{\Gamma}(h_0)$ can be solved by the recursive paradigm as described above. For each $h_{t-1} \in H_{t-1}$ such that $\bar{\mu}(h_{t-1}) = \mu_{t-1}$, a profile $\bar{\gamma}(h_{t-1}) = (\pi_t, \{\bar{\gamma}(h_{t-1}, \pi_t, s)\}_{s\in S})$ is an SPE path of $G(h_{t-1})$ if and only if $\bar{v}^t(h_{t-1}; \bar{\gamma}(h_{t-1})) \geq \text{cl}(\underline{V}_t^t)(\mu_{t-1})$. An SPE always exists in $G(h_{t-1})$. Furthermore, V_t^k is nonempty valued and has closed graph for each k and t.*

7 Constant-Sum Utilities and Full Revelation

In the literature of communication games, it has been extensively discussed the relationship between conflicts of interest and information revelation. Milgrom and Roberts (1986) have shown that if the receiver has advisors with conflicting interest, then full revelation can be sustained even when the receiver is unsophisticated and ill-informed about the preferences of the senders. In the sequential cheap talk model studied by Krishna and Morgan (2001), the receiver is able to extract more information from senders who have opposing biases, but she will not have full information unless she add another stage for senders to rebuke each other. Battaglini (2002) obtains a strong result that full information is generically possible regardless of incentives once extending the cheap talk model to allow for multidimensional states and more than one senders. Ravindran and Cui (2020) also analyze competition among senders in the context of Bayesian persuasion. Unlike this paper, in their setup, there are two or more senders who move simultaneously and employ independent signal structures. They conclude that full information is the unique equilibrium outcome if and only if senders' utilities are *sufficiently nonlinear*.

Here I present a result that embodies the idea that competition among senders can give rise to full information in the context of Bayesian persuasion.

Theorem 2. *In a sequential persuasion game with $T \geq 2$ senders, if there are two senders who have constant-sum utilities, full revelation is supported as an equilibrium outcome.*

	L	C	R
ω_0	$(4,0,3)$	$(3,3,2)$	$(0,4,0)$
ω_1	$(4,0,0)$	$(3,3,2)$	$(0,4,3)$

Table 2: Two Senders with Conflicting Interest

In this paragraph I discuss the intuition of the proof when $T=2$. First, from Theorem 1, we know that after any history $h_1 \in H_1$ such that $\bar{\mu}(h_1)=\mu_1$, the most favorable equilibrium outcome for sender 2 results in a payoff $\bar{V}_1^2(\mu_1)=\text{cl}(\bar{V}_2^2)(\mu_1)$. Because the senders have constant-sum utilities (assuming they sum to $c \in \mathbb{R}$), the best SPE outcome for sender 2 turns out to be the least favorable for sender 1, so that $\underline{V}_1^1 = c - \bar{V}_1^2$. Note that $\bar{V}_1^2$ is a concave function in μ_1, which means $\underline{V}_1^1$ is convex, and the criterion for sender 1's equilibrium strategy $\text{cl}(\underline{V}_1^1)$ is a hyperplane. This hyperplane coincides with the continuation payoffs for sender 1 deriving from fully revealing the states. By Theorem 1, it is optimal for sender 1 to fully reveal the state.

Nevertheless, the full information result is sensitive to the assumption of constant-sum utilities. A plausible alternative condition on the payoff structure is conflicting interests, which also implies that senders have opposite ordinal preferences over outcomes. But there is a counter example to this conjecture as below.

Suppose in a game with two senders, $\Omega=\{\omega_0,\omega_1\}$, $A=\{L,C,R\}$, and $\mu_0(\omega_1)=\frac{2}{3}$. Each entry of Table 2 presents the payoffs to sender 1, sender 2, and the receiver. Note that senders' payoffs are independent of the state, sender 1 prefers L to C to R, while sender 2 has the opposite ordinal preferences. The receiver would like to take L when $\mu(\omega_1) \leq \frac{1}{3}$, C when $\frac{1}{3} \leq \mu(\omega_1) \leq \frac{2}{3}$, and R when $\mu(\omega_1) \geq \frac{2}{3}$. Figure 3 depicts senders' value correspondences V_2^1 and V_2^2 as the black lines.

Let me operate the recursive method to solve for the equilibrium outcome of this game, as illustrated by Figure 3. The red curve in the lower figure indicates sender 2's value function V_1^2, which is the concave closure of V_2^2; accordingly, and the red curve in the upper figure indicates V_1^1. Then, I concavify V_1^1 and obtain the blue curve in the upper figure. This function illustrates sender's 1 equilibrium payoff. It is easy to see that when $\mu_0(\omega_1)=\frac{2}{3}$, there is only one optimal way for sender 1 to provide persuasion, that is, to induce an information policy supported on $\{\frac{1}{3},1\}$. Thus, in the unique equilibrium, the sender only reveals partial information.

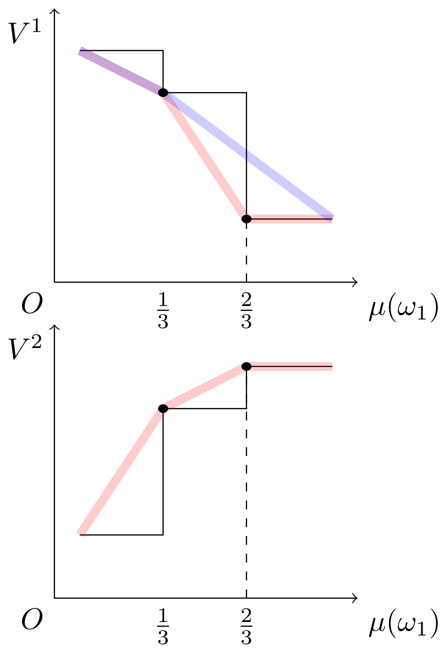

Figure 3: The equilibrium in the case of conflicting interest

8 Silent Equilibrium

When senders move simultaneously and senders can correlate their signal structures (Gentzkow and Kamenica, 2017a,b), it suffices to let only one sender reveal information on the equilibrium path. But in the sequential case this may not be true. Because later senders' strategies are contingent on those of prior senders, it is possible that the second sender always wants to add something to what the first sender has told. However, I find that there generally exists a specific type of equilibrium where only one sender "speaks," which I call a *silent* equilibrium.

Definition 3. *An SPE is a silent equilibrium if at most one sender reveals information on the equilibrium path.*

It may be counter-intuitive at first glance why senders are willing to waste their opportunities to persuade. The reason is in large part due to sequential moves and public signals. Under any existing signal structure, the next sender is able to come up with another signal structure if and only if it is weakly more informative in Blackwell order. In a silent equilibrium, senders incorporate follow-up persuasion into their own persuasion, and then there is no need for subsequent senders to take further actions, given that what they would like to "say" has already been told. On the other hand, if they deviate, either their speeches make no difference, or subsequent senders will reveal information in a more harmful way.

In what follows, I will first prove the existence of a silent equilibrium and then give a condition for the outcome equivalence between SPE and silent equilibria.

8.1 Existence

Theorem 3. *There exists a silent equilibrium.*

The proof is constructive. In the construction, the strategy of each player prescribes the same action contingent on the set of histories associated with the same belief. To abuse the notion, I write $\sigma_t^M(\mu)$ and $\rho^M(\mu)$ as sender t's strategy and receiver's strategy in the silent equilibrium. Also, in this subsection I define $V_t^k(\mu) = \{\bar{v}^k(\sigma_{t+1}^M, \dots, \sigma_T^M, \rho^M) \mid \mu_t = \mu\}$ as senders' value correspondences based on the equilibrium strategies I construct. The contingency on beliefs suggests that V_t^k is a single-valued function.

Consider the receiver first. When she is indifferent between multiple actions, she breaks ties in a way that favors senders who comes later with descending priority: sender $T \succ$ sender $(T-1) \cdots \succ$ sender 1. Find any optimal ρ^M that satisfies this tie-breaking rule. Given such ρ^M, sender T has an upper semi-continuous (henceforth USC) value function V_T^T. Then I can solve for sender T's optimal strategy σ_T^M (Kamenica and Gentzkow, 2011).

Let me introduce an important concept, the *silent* set for sender T, $\mathcal{S}^T = \{\mu_{T-1} \in \Delta(\Omega) \mid V_T^T(\mu_{T-1}) = \text{cl}(V_T^T)(\mu_{T-1})\}$. $\mathcal{S}^T$ coincides with the set of beliefs such that sender T does not benefit from providing additional information. When $\mu_{T-1} \in \mathcal{S}^T$, let Sender T babble; When $\mu_{T-1} \notin \mathcal{S}^T$, let sender T spread posteriors over the silent set, i.e., $\text{supp}(\tau_T) \subseteq \mathcal{S}^T$.

Based on Sender T's behavior, Sender $T-1$ realizes that the induced belief will finally lie in $\mathcal{S}^T$. Therefore, he only needs to consider $\mathcal{S}^T$ as the feasible support of his information policy. It can be proved that $\mathcal{S}^T$ is closed and by the receiver's tie-breaking rule, V_{T-1}^{T-1} is upper semi-continuous on $\mathcal{S}^T$. So, there exists an optimal strategy for Sender $T-1$, σ_{T-1}^M.

Recursively, I can obtain senders' optimal strategies, value functions, and silent sets $\mathcal{S}^t = \{\mu_{t-1} \in \Delta(\Omega) \mid V_t^t(\mu_{t-1}) = \text{cl}(V_t^t(\mu_{t-1}))\} \cap \mathcal{S}^{t+1}$. There exists an optimal strategy for each sender because $\mathcal{S}^{t+1}$ is closed and V_t^t is upper semi-continuous on $\mathcal{S}^{t+1}$. By construction, the silent sets are shrinking as t decreases. After Sender 1 spreads his posteriors over $\mathcal{S}^1$, it holds that $\mu_1 \in \mathcal{S}^t$, for all $t = 2, \dots, T$, so that all the subsequent senders are willing to stay silent.

8.2 Outcome equivalence between SPE and silent equilibria

Heuristically, any equilibrium outcome should be supported in an equilibrium where the first sender reveals all the information the subsequent senders would have revealed on the equilibrium path. However, this is not true. For senders who come later, to voluntarily provide a piece of information is different from taking it as given and saying nothing. To accept the existing signal structure, it requires that, *for each signal realized*, sender t should receive high enough payoff from keeping silent. But if it is sender t himself who reveals information, he only needs to receive high enough *expected* payoff from sending all those signals. In the second case, the incentive constraints for the equilibrium could be relaxed. In Appendix D.1, I will present an SPE outcome that requires more than one sender to reveal information.

Nevertheless, I find that, for each sender t, the difference between revealing information by himself and by previous senders disappears when the concave closures over his maximal and minimal value function coincide, i.e., $\mathrm{cl}(\bar{V}_t^t) = \mathrm{cl}(\underline{V}_t^t)$. This is a sufficient condition for the outcome equivalence between SPE and silent equilibria. There are many examples that satisfy this condition, including the motivating examples in Kamenica and Gentzkow (2011) and Board and Lu (2018), the court example at the beginning of Section 1, and the conflicting interest game in Section 7. Below, I will formalize this result.

Let me define an equilibrium outcome as the *ex ante* distribution of posteriors induced from an SPE and the associated (mixed) actions with each posterior.[3] Suppose there is an SPE $(\sigma_1, \dots, \sigma_T, \rho)$. Let (τ^e, λ^e) be the equilibrium outcome. τ^e is the *ex ante* distribution of posterior beliefs deriving from the signal structures on the equilibrium path. λ^e is the function that maps each equilibrium posterior belief to the induced (mixed) action. For each $\mu \in \mathrm{supp}(\tau^e)$, there is a signal vector $s \in \prod_{t=1}^T S_t$ such that the posterior belief $\bar{\mu}(s) = \mu$ and the associated action $\lambda^e(\mu) = \rho(s)$.

Definition 4. *Two SPE are* outcome equivalent *if their equilibrium outcomes, (τ_1^e, λ_1^e) and (τ_2^e, λ_2^e), are the same.*

Finally, I present the theorem that shows the equivalence result.

Theorem 4. *If for each $t \geq 2$, $\mathrm{cl}(\bar{V}_t^t) = \mathrm{cl}(\underline{V}_t^t)$, every outcome of an SPE is supported by a silent equilibrium.*

[3]This definition is closely related to the standard definition of equilibrium outcome as induced distribution over the product space of states and actions, for that an equilibrium outcome under my definition refers to a unique equilibrium outcome under the latter definition.

It is worth noting that this result does not hold for Markov Perfect Equilibrium (MPE), where I contingent strategies on beliefs as in Section 8.1. Under an MPE, the value correspondence of each sender is a single-valued function in belief, with a unique concave closure. It seems plausible that it is a case that satisfies the condition of Theorem 4. However, to let subsequent senders babble on the MPE path no only changes the on-path behavior, but also the off-path behavior conditional on some realized belief. As a result, it will further affect preceding senders' optimal choices and the silent equilibrium might break down. By contrast, an SPE is immune to this effect. A strategy contingent on histories can differentiate between information from different sources, despite the same posterior realized. So the value correspondences are invariant to whichever SPE path is considered.

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Appendix

A Section 5

A.1

Proposition 2. *V is nonempty valued and has closed graph.*

Proof. By the Maximum Theorem, $r(\cdot)$ has nonempty and compact values for each $\mu \in \Delta(\Omega)$ and is upper hemicontinuous (uhc). Thus, V is nonempty valued.

Suppose there is a sequence (μ^n, v^n) such that $v^n \in V(\mu^n)$ for each $n \in \mathbb{N}$ and $(\mu^n, v^n) \to (\mu, v^*)$. For each $n \in \mathbb{N}$, there is $\alpha^n \in r(\mu^n)$ such that $v^n = \mathbb{E}[v(\alpha^n, \omega) \,|\, \mu^n]$. Because r is uhc, $(\alpha^n)_{n=1}^{\infty}$ has a limit point $\alpha \in r(\mu)$. Then, because v is continuous, $v^* = \mathbb{E}[v(\alpha, \omega) \,|\, \mu] \in V(\mu)$. □

A.2 Proof of Lemma 1

Proof. The proof is straightforward after laying out $\mathrm{cl}(\underline{V})$ algebraically according to Definition 2.

$$
\begin{aligned}
\mathrm{cl}(\underline{V})(\mu_0) &= \sup_{\substack{\tau\in\Delta(\Delta(\Omega))\\ \mathbb{E}[\tau]=\mu_0}} \sum_{\mu\in\mathrm{supp}(\tau)} \tau(\mu)\cdot\underline{V}(\mu) \\
&= \sup_{\substack{\tau\in\Delta(\Delta(\Omega))\\ \mathbb{E}[\tau]=\mu_0}} \sum_{\mu\in\mathrm{supp}(\tau)} \tau(\mu)\cdot \min_{\alpha\in r(\mu)} \mathbb{E}[v(\alpha,\omega)\,|\,\mu]
\end{aligned}
\tag{3}
$$

If $\bar{v}(\pi,\bar{\alpha}) \geq \mathrm{cl}(\underline{V})$, I can construct a SPE (σ,ρ) that is consistent with $(\pi,\bar{\alpha})$. Let $\sigma(\mu_0)=\pi$ and ρ coincide with $\bar{\alpha}$ on the equilibrium path. When the sender deviates to any other signal structure π', ρ prescribes a *minimal* reaction for each $h=(\pi',s)$ such that

$$
\rho(h) \in \operatorname*{arg\,min}_{\alpha\in r(\bar{\mu}(h))} \mathbb{E}[v(\alpha,\omega)\,|\,\bar{\mu}(h)] \tag{4}
$$

In other words, if the sender deviates from π, the receiver will break ties by punishing him as severe as possible. Given ρ, the *ex ante* payoff for the sender when he deviates to π' (which induces τ') can not exceed the concave closure of the minimal value.

$$
\bar{v}(\pi',\rho) = \sum_{\mu\in\mathrm{supp}(\tau')} \tau'(\mu)\cdot \min_{\alpha\in r(\mu)} \mathbb{E}[v(\alpha,\omega)\,|\,\mu] \leq \mathrm{cl}(\underline{V})(\mu_0) \leq \bar{v}(\pi,\rho)
$$

On the other hand, suppose (σ,ρ) is an equilibrium consistent with $(\pi,\bar{\alpha})$ and $\bar{v}(\pi,\bar{\alpha}) < \mathrm{cl}(\underline{V})(\mu_0)$, then there exists a signal structure π' (inducing τ') that gives the sender an *ex ante* payoff higher than $\bar{v}(\pi,\bar{\alpha})$ even if the receiver takes minimal actions given by Eq. (3). Therefore, it is profitable for the sender to deviate to π' because

$$
\bar{v}(\pi',\rho) = \sum_{\mu\in\mathrm{supp}(\tau')} \tau'(\mu)\cdot\mathbb{E}[v(a,\omega)\,|\,\rho,\mu] \geq \sum_{\mu\in\mathrm{supp}(\tau')} \tau'(\mu)\cdot\underline{V}(\mu) > \bar{v}(\pi,\rho)
$$

This is a contradiction to that $(\pi,\bar{\alpha})$ is an equilibrium path. □

A.3 Proof of Lemma 2

Proof. Because V has closed graph, $\bar{V}$ is a well defined upper-semi continuous function on $\Delta(\Omega)$. There is a path $(\pi,\bar{\alpha})$, where π induces τ, such that: (1) for each $s\in S$, $\bar{\alpha}(s)\in r(\bar{\mu}(\pi,s))$ and $\mathbb{E}[v(\bar{\alpha}(s),\omega)\,|\,\mu]=\bar{V}(\mu)$; (2) $\bar{v}(\pi,\bar{\alpha})=\mathrm{cl}(\bar{V})(\mu_0)$.

Since $\mathrm{cl}(\bar{V})(\mu_0)\geq\mathrm{cl}(\underline{V})(\mu_0)$, by Lemma 1, $(\pi,\bar{\alpha})$ is an equilibrium path. □

A.4 Proof of Proposition 1

Suppose $v^* = (1-\beta)\,\mathrm{cl}(\underline{V})(\mu_0) + \beta\,\mathrm{cl}(\bar{V})(\mu_0)$, for $\beta \in [0,1]$. I want to show that v^* is an equilibrium payoff for the sender.

Because the best response r is upper hemicontinuous and V has closed graph, $\mathrm{cl}(\underline{V})(\mu_0)$ and $\mathrm{cl}(\bar{V})(\mu_0)$ are equilibrium payoffs via paths $\{\underline{\tau}, \{\underline{\alpha}(s_1)\}_{s_1 \in S_1}\}$ and $\{\bar{\tau}, \{\bar{\alpha}(s_2)\}_{s_2 \in S_2}\}$. Then consider another path

$$\{(1-\beta)\underline{\tau} + \beta\bar{\tau}, \quad \{\underline{\alpha}(s)\}_{s \in S_1 \setminus S_2} \bigcup \{\bar{\alpha}(s)\}_{s \in S_2 \setminus S_1} \bigcup \{\alpha^*(s)\}_{s \in S_1 \cap S_2}\},$$

where $\alpha^*(s) = (1-\beta)\underline{\alpha}(s) + \beta\bar{\alpha}(s)$. On this path, the receiver best responds to each signal and the sender's payoff is equal to $v^* \geq \mathrm{cl}(\underline{V})(\mu_0)$. According to Lemma 1, This is an equilibrium path.

B Section 6

B.1 Proof of Theorem 1

Throughout this section, I restrict the signal space S to containing $(N+T)$ elements. Heuristically, this will narrow down the ranges of $\{V_t^k\}_{k=1}^T$. But it is not the case. From a geometric point of view, each point in the graph of V_t^k must be the convex combination of the points inside the graph of V_{t+1}^k prescribed by σ_{t+1}. By Carathéodory theorem, any point in $graph\{(V_{t+1}^1, V_{t+1}^2, \ldots, V_{t+1}^T)\}$ can be expressed as the convex combination of at most $(l+T)$ points in $graph\{(V_t^1, V_t^2, \ldots, V_t^T)\}$, in that $graph\{(V_t^1, V_t^2, \ldots, V_t^T)\}$ is of $(N+T-1)$ dimensions. It suffices to use a signal structure that sends at most $(N+T)$ signals.

Also, the finite state space implies that the set of signal structures $\Pi = (\Delta S)^\Omega$ is compact. For any history $h_t \in H_t$, an equilibrium path $\bar{\gamma}(h_t)$ composes of a sequence of contingent signal structures and a distribution of actions, $(\pi_{t+1}, \pi_{t+2}(\cdot|s_{t+1}), \ldots,$ $\pi_T(\cdot|s_{T-1}, \ldots, s_{t+1})$, $a(s_T, \ldots, s_{t+1}))_{(s_T, \ldots, s_{t+1}) \in S^{T-t}}$, which is contained in a compact set $\Pi \times \Pi^N \times \cdots \times \Pi^{N^{T-t-1}} \times A^{N^{T-t}}$. So any sequence of the equilibrium paths has a converging subsequence. Roughly speaking, this property, combined with the continuous utility functions, leads to the closedness of value correspondence.

Before diving into the details, I present two useful lemmas concerned with how converging signal structures give rise to converging beliefs and utilities.

Lemma 3. *For any $\mu^n \in \Delta\Omega$, $\pi^n \in \Pi$ such that $\mu^n \to \mu$ and $\pi^n \to \pi$, let $\lambda^n(\cdot|s)$, $\lambda(\cdot|s)$ represent posterior beliefs induced by a signal s under π^n, π. Then, for any ω and any $s \in \bigcup_{\omega \in \Omega} \mathrm{supp}(\pi(\omega))$, $Pr(s|\pi, \mu) = \lim_{n \to \infty} Pr(s|\pi^n, \mu^n)$ and $\lambda(\omega|s) = \lim_{n \to} \lambda^n(\omega|s)$.*

Proof. For any $s \in S$,

$$Pr(s|\pi,\mu) = \sum_{\omega\in\Omega} \mu(\omega)\cdot\pi(s|\omega) = \lim_{n\to\infty}\sum_{\omega\in\Omega}\mu^n(\omega)\pi^n(s|\omega) = \lim_{n\to\infty} Pr(s|\pi^n,\mu^n)$$

For any $s \in \bigcup_{\omega\in\Omega}\text{supp}(\pi(\omega))$ and any $\omega \in \Omega$,

$$\lambda(\omega|s) = \frac{\pi(s|\omega)\mu(\omega)}{\sum_{\omega'\in\Omega}\pi(s|\omega')\mu(\omega')} = \lim_{n\to\infty}\frac{\pi^n(s|\omega)\mu^n(\omega)}{\sum_{\omega'\in\Omega}\pi^n(s|\omega')\mu^n(\omega')} = \lim_{n\to\infty}\lambda^n(\omega|s)$$

□

Lemma 4. *For any $\{(h^n)_{n=1}^{\infty}, h\} \subseteq H_t$ s.t. $\bar{\mu}(h^n) = \mu^n$, $\bar{\mu}(h) = \mu$, and $\mu^n \to \mu$. Suppose there is a sequence of paths $\gamma^n(h^n)$ that converges to a path $\gamma(h)$, then for any $k = 1, \ldots, T$, $\bar{v}^k(h^n;\gamma^n(h^n)) \to \bar{v}^k(h;\gamma(h))$.*

Proof.

$$\begin{aligned}
&\lim_{n\to\infty}\bar{v}^k(h^n;\gamma^n(h^n))\\
=&\lim_{n\to\infty}\sum_{\omega\in\Omega}\sum_{s\in S^{T-t}}\mu^n(\omega)\cdot\pi^n_{t+1}(s_{t+1}|\omega)\cdots\pi^n_T(s_T|\omega)\cdot v^k(a^n(s),\omega)\\
=&\sum_{\omega\in\Omega}\sum_{s\in S^{T-t}}\mu(\omega)\cdot\pi_{t+1}(s_{t+1}|\omega)\cdots\pi_T(s_T|\omega)\cdot v^k(a(s),\omega)\\
=&\bar{v}^k(h;\gamma(h))
\end{aligned}$$

□

Based on Lemmas 3 and 4, I can conduct an induction that demonstrate Theorem 1. Suppose

1. For any $j \geq t$, $h_j \in H_j$, $\bar{\Gamma}(h_t)$ can be solved.
2. For any k and $j \geq t+1$, V_j^k is nonempty valued and has closed graph.
3. For any $\{h^n\}, h \in H_t$ such that $\bar{\mu}(h^n) = \mu^n$, $\bar{\mu}(h) = \mu_t$, and $\mu^n \to \mu_t$. Suppose there is a sequence of equilibrium paths $\{\bar{\gamma}(h^n)\}$ that converges to a path $\gamma(h)$. Then, $\gamma(h) \in \bar{\Gamma}(h)$.

Then,

Proposition 3. *For any k, $\{V_t^k\}_{k=1}^T$ have closed graph.*

Proof. For any $\mu^n \to \mu_t$ and $v^n \to v$ such that $v^n \in V_t^k(\mu^n)$ for $n \in \mathbb{N}$, there are histories $h^n, h \in H_t$ such that $\bar{\mu}(h^n) = \mu^n$ and $\bar{\mu}(h) = \mu_t$; also, there are equilibrium paths $\bar{\gamma}(h^n)$ such that $\bar{v}^k(h^n; \bar{\gamma}(h^n)) = v^n$.

As have been argued before, there is a converging subsequence $\{\bar{\gamma}(h^{n_i})\} \to \gamma(h)$. By induction (3), $\gamma(h) \in \bar{\Gamma}(h)$, so that $\bar{v}(h; \bar{\gamma}(h)) \in V_t^k(\mu)$. By Lemma 3,

$$v = \lim_{i \to \infty} v^{n_i} = \lim_{i \to \infty} \bar{v}^k(h^{n_i}; \bar{\gamma}(h^{n_i})) = \bar{v}(h; \bar{\gamma}(h))$$

□

Proposition 4. *For any $h_{t-1} \in H_{t-1}$, $\bar{\mu}(h_{t-1}) = \mu_{t-1}$, a path $\gamma(h_{t-1}) = (\pi_t, \{\bar{\gamma}(h_{t-1}, \pi_t, s)\}_{s \in S})$ is an equilibrium path if and only if $\bar{v}^t(h_{t-1}; \gamma(h_{t-1})) \geq \mathrm{cl}(\underline{V}_t^t)(\mu_{t-1})$.*

Proof. For any $h_t \in H_t$, there is an equilibrium path that gives sender t the minimal payoff $\underline{\gamma}(h_t)$, i.e., $\bar{v}^t(h_t; \underline{\gamma}(h_t)) = \underline{V}_t^t(\bar{\mu}(h_t))$. Let $\bar{s}(\mu)$ be the signal inducing μ.[4]

By Definition 2,

$$\mathrm{cl}(\underline{V}_t^t)(\mu_{t-1}) = \sup_{\substack{\tau_t \in \Delta(\Delta(\Omega)) \\ E[\tau] = \mu_{t-1}}} \sum_{\mu \in \mathrm{supp}(\tau_t)} \tau_t(\mu) \cdot \bar{v}^t(h_{t-1}; \underline{\gamma}(h_{t-1}; \pi_t, \bar{s}(\mu))$$

(Sufficiency) I want to show that a path $\gamma(h_{t-1})$ that satisfies the condition is an SPE path. Construct a strategy profile that prescribes $\gamma(h_{t-1})$ on the path, and if sender t deviates from π (τ) to π' (τ'), it is followed by an equilibrium path $\underline{\gamma}(h_t)$ for any $h_t \in H_t$. Therefore, the equilibrium payoff for sender t is no less than that from any deviation.

$$\bar{v}^t(h_{t-1}; \gamma(h_{t-1})) \geq \mathrm{cl}(\underline{V}_t^t)(\mu_{t-1}) \geq \sum_{\mu \in \mathrm{supp}(\tau_t')} \tau_t'(\mu) \cdot \bar{v}^t(h_{t-1}; \underline{\gamma}(h_{t-1}; \pi_t', \bar{s}(\mu))$$

(Necessity) If $\bar{v}^t(h_{t-1}; \gamma(h_{t-1})) < \mathrm{cl}(\underline{V}_t^t)(\mu_{t-1})$, that means there exists π_t' (τ_t') such that

$$\bar{v}^t(h_{t-1}; \gamma(h_{t-1})) < \sum_{\mu \in \mathrm{supp}(\tau_t')} \tau_t'(\mu) \cdot \bar{v}^t(h_{t-1}; \underline{\gamma}(h_{t-1}, \pi_t', \bar{s}(\mu)) \tag{5}$$

Suppose any strategy profile $(\sigma_t, \sigma_{t+1}, \ldots, \rho)$ that forms an equilibrium and yields $\gamma(h_{t-1})$. If sender t deviates to π_t', he would obtain a payoff

$$\sum_{\mu \in \mathrm{supp}(\tau_t')} \tau_t'(\mu) \cdot \bar{v}^t(h_{t-1}; \pi_t', \bar{s}(\mu) | \sigma_{t+1}, \ldots, \rho) \geq \sum_{\mu \in \mathrm{supp}(\tau_t')} \tau_t'(\mu) \cdot \bar{v}^t(h_{t-1}; \underline{\gamma}(h_{t-1}, \pi_t', \bar{s}(\mu))$$

[4]When μ is induced by two signals under a signal structure, it is equivalent to study another signal structure that induce μ with only one signal.

(6)

Combining Eq. (5) and (6), apparently that it is profitable for sender t to deviate to π'. Contradiction.

$\square$

Proposition 5. *For any $h_{t-1} \in H_{t-1}$, there exists an SPE in $G(h_{t-1})$.*

Proof. Let $\bar{\mu}(h_{t-1}) = \mu_{t-1}$. By Proposition 3, $\bar{V}_t^t$ is a well defined upper semi-continuous function on $\Delta\Omega$. There is a path $(\pi_t, \{\bar{\gamma}(h_{t-1}, \pi_t, s)\}_{s \in S})$, where π_t induces τ_t, such that: (1) for any $\mu \in \text{supp}(\tau_t)$, $\bar{\gamma}(h_{t-1}, \pi_t, \bar{s}(\mu)) \in \bar{\Gamma}(h_{t-1}, \pi_t, \bar{s}(\mu))$ and $\bar{v}^t(h_{t-1}; \bar{\gamma}(h_{t-1}; \pi_t, \bar{s}(\mu)) = \bar{V}_t^t(\mu)$; (2)

$$\sum_{\mu \in \text{supp}(\tau_t)} \tau_t(\mu) \cdot \bar{v}^t(h_{t-1}; \bar{\gamma}(h_{t-1}; \pi_t, \bar{s}(\mu)) = \text{cl}(\bar{V}_t^t)(\mu_{t-1}) \geq \text{cl}(\underline{V}_t^t)(\mu_{t-1})$$

By Proposition 4, $(\pi_t, \{\bar{\gamma}(h_{t-1}, \pi_t, s)\}_{s \in S})$ is an equilibrium path.

$\square$

Proposition 6. *For any $\{(h^n)_{n=1}^{\infty}, h\} \in H_{t-1}$, such that $\bar{\mu}(h^n) = \mu^n$, $\bar{\mu}(h) = \mu_{t-1}$, and $\mu^n \to \mu_{t-1}$. Suppose there is a sequence of equilibrium paths $\bar{\gamma}(h^n)$ that converges to a path $\gamma(h)$. Then $\gamma(h) \in \bar{\Gamma}(h)$.*

Proof. Because $\pi_t^n \to \pi_t$, by Lemma 3, for any $s \in S$, $\bar{\mu}(h^n, \pi_t^n, s) \to \bar{\mu}(h, \pi_t, s)$ and $\bar{\gamma}(h^n, \pi_t^n, s) \in \bar{\Gamma}(h^n, \pi_t^n, s)$, so $\bar{\gamma}(h, \pi_t, s) \in \bar{\Gamma}(h, \pi_t, s)$ by induction (3).

By Proposition 4, $\bar{v}^t(h^n; \bar{\gamma}^n(h^n)) \geq \text{cl}(\underline{V}_t^t)(\mu^n)$.

By Lemma 4 and that the concave function $\text{cl}(\underline{V}_t^t)$ is continuous on $\Delta\Omega$,

$$\begin{aligned} \bar{v}^t(h; \gamma(h)) &= \lim_{n \to \infty} \bar{v}^t(h^n; \bar{\gamma}^n(h^n)) \\ &\geq \lim_{n \to \infty} \text{cl}(\underline{V}_t^t)(\mu^n) \\ &= \text{cl}(\underline{V}_t^t)(\mu) \end{aligned}$$

By Proposition 4, $\gamma(h) \in \bar{\Gamma}(h_{t-1})$. $\square$

Finally, Propositions 3 - 6 conclude the proof of Theorem 1.

C Section 7

C.1 Proof of Theorem 2

Suppose the two senders who have constant-sum utilities are senders i and j $(i<j)$. Specifically, $v^i+v^j=c$, where $c\in\mathbb{R}$. By definition, an equilibrium in $\Gamma(h_t)$ that leads to the maximum payoff for i would generate the minimum payoff for j. That is, for any t, $\bar{V}_t^i+\underline{V}_t^j=c$, for each t.

According to Theorem 1, V_{j-1}^j is bounded below by the concave closure of the minimal function $\underline{V}_j^j$. Because $\bar{V}_{j-1}^i+\underline{V}_{j-1}^j=c$, $\bar{V}_{j-1}^i$ is bounded above by the convex function $c-\text{cl}(\underline{V}_j^j)$.

At each extreme point $\delta(\omega)\in\Delta(\Omega)$, $c-\text{cl}(\underline{V}_j^j)(\delta(\omega))=c-\underline{V}_j^j(\delta(\omega))=\bar{V}_j^i(\delta(\omega))$. Because of convexity, $c-\text{cl}(\underline{V}_j^j)$ is dominated by the hyperplane $f:\Delta(\Omega)\longrightarrow\mathbb{R}$ such that $f(\delta(\omega))=\bar{V}_j^i(\delta(\omega))$, for each ω. Mathematically, $f(\mu)=\sum_{\omega\in\Omega}\mu(\omega)\bar{V}_j^i(\delta(\omega))$.

Let hypo$(\cdot)$ denote the hypograph of a function. From above, graph$(V_{j-1}^i)\subseteq$ hypo(f). Furthermore, since f is linear, hypo(f) is a convex set. In addition, graph(V_{j-2}^i) consists of points that are convex combinations of points in graph(V_{j-1}^i). So graph$(V_{j-2}^i)\subseteq$ hypo(f). By similar arguments, graph$(V_t^i)\subseteq$ hypo(f), for each $t\leq j-1$.

Note that no matter in which period the true state is revealed, the remaining rounds are irrelevant for the decision making. That means the value correspondences are invariant at the extremes throughout the recursion, that is, $V_t^k(\delta(\omega))=V_{t'}^k(\delta(\omega))$, $\forall k,\forall t\neq t',\forall\omega$. As a result, I can also write $f(\mu)=\sum_{\omega\in\Omega}\mu(\omega)\bar{V}_i^i(\delta(\omega))$.

For any $\mu\in\Delta(\Omega)$, if $v^*\in V_{i-1}^i(\mu)$, there must be a mean-preserving spread τ of μ, such that for each $\mu'\in\text{supp}(\tau)$, there is $v(\mu')\in V_i^i(\mu')$ that satisfies $v^*=\sum_{\mu'\in\text{supp}(\tau)}\tau(\mu')\cdot v(\mu')\leq\sum_{\mu'\in\text{supp}(\tau)}\tau(\mu')\cdot f(\mu')=f(\mu)$. The inequality derives from that graph$(V_i^i)\subseteq$ hypo(f). The last equation derives from that f is linear.

By Theorem 1, $\text{cl}(\underline{V}_i^i)(\mu)\leq v^*\leq f(\mu)$. Because $f(\mu)$ can be achieved by sender i fully revealing information and the receiver taking subsequent actions that lead to $\bar{V}_i^i(\delta(\omega))$, it is optimal for sender i to fully reveal information. When there is at least one sender telling the truth, the receiver would know the true state at the end.

D Section 8

D.1 An SPE outcome that needs two players to reveal information

The state space is $\{\omega_1,\omega_2\}$ and both states are equally likely. There are two senders and one receiver. The receiver has an action space $\{a_1,\dots,a_8\}$. The players' payoffs

Action

		a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
State	ω_1	0,1,7	4,1,6	0,16,4	0,4,1	0,5,1	0,-5,-3	4,1,-8	0,1,-14
	ω_2	0,1,-14	4,1,-8	0,-5,-3	0,4,1	0,5,1	0,16,4	4,1,6	0,1,7

Table 3: Payoff Table

depend on the state and action, which are shown in Table 3. In each table entry, there are payoffs for sender 1, sender 2, and the receiver, respectively. Let μ_t denote the posterior probability of ω_1 after period t and $r(\mu_2)$ the receiver's best responses to μ_2. When $\mu_2 \leq \frac{1}{7}$, $r(\mu_2) = \{a_1\}$; when $\frac{1}{7} \leq \mu_2 \leq \frac{2}{7}$, $r(\mu_2) = \{a_2\}$; when $\frac{2}{7} \leq \mu_2 \leq \frac{3}{7}$, $r(\mu_2) = \{a_3\}$; when $\frac{3}{7} \leq \mu_2 \leq \frac{4}{7}$, $r(\mu_2) = \{a_4, a_5\}$; when $\frac{4}{7} \leq \mu_2 \leq \frac{5}{7}$, $r(\mu_2) = \{a_6\}$; when $\frac{5}{7} \leq \mu_2 \leq \frac{6}{7}$, $r(\mu_2) = \{a_7\}$; when $\frac{6}{7} \leq \mu_2 \leq 1$, $r(\mu_2) = \{a_8\}$. Based on the receiver's best responses, I plot sender 2's continuation payoff correspondence, as shown in Figure 4.

Next, I will present an SPE outcome where sender 1 gets an expected payoff 2 and then show that this equilibrium outcome cannot be achieved without two players revealing a certain amount of information. The strategy profile that forms the SPE path is illustrated as below.

$$\pi_1(s_1|\omega_1) = \frac{2}{7} \qquad \pi_1(s_2|\omega_1) = \frac{5}{7}$$
$$\pi_1(s_1|\omega_2) = \frac{5}{7} \qquad \pi_1(s_2|\omega_2) = \frac{2}{7}$$

Under π_1, there are probabilities 50-50 that the signal sent is s_1 or s_2, where s_1 updates the belief to $\mu_1 = \frac{2}{7}$ and s_2 to $\mu_1 = \frac{5}{7}$. For $i = 1, 2$, let $\pi_{2,i}$ be the signal structure sender 2 designs after seeing s_i. Then, $\pi_{2,1}$ satisfies that

$$\pi_{2,1}(\hat{s}_1|\omega_1) = \frac{1}{4} \qquad \pi_{2,1}(\hat{s}_2|\omega_1) = \frac{3}{4}$$
$$\pi_{2,1}(\hat{s}_1|\omega_2) = \frac{3}{5} \qquad \pi_{2,1}(\hat{s}_2|\omega_2) = \frac{2}{5}$$

And $\pi_{2,2}$ satisfies that

$$\pi_{2,2}(\hat{s}_1|\omega_1) = \frac{2}{5} \qquad \pi_{2,2}(\hat{s}_2|\omega_1) = \frac{3}{5}$$
$$\pi_{2,2}(\hat{s}_1|\omega_2) = \frac{3}{4} \qquad \pi_{2,2}(\hat{s}_2|\omega_2) = \frac{1}{4}$$

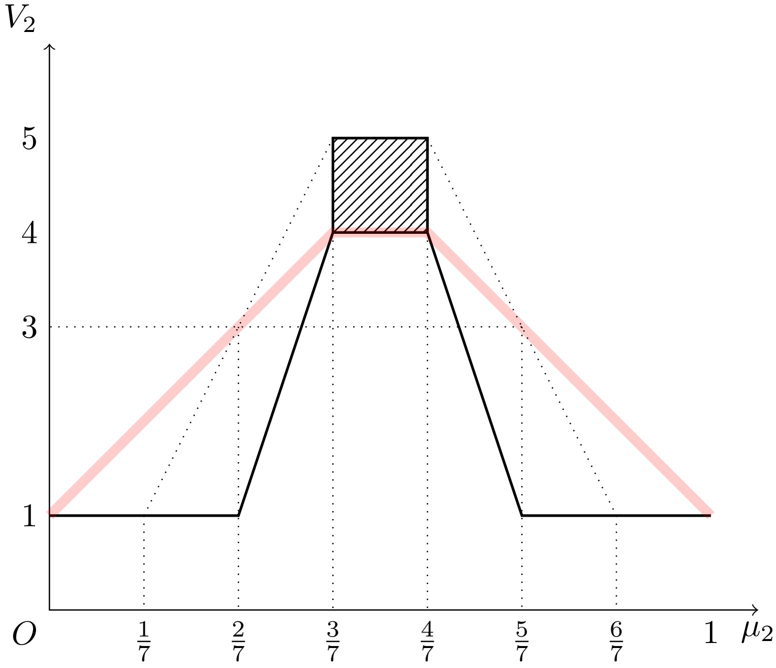

Figure 4: Sender 2's continuation payoff correspondences on μ_2

Under this signal structure, the belief is updated to $\mu_2 = \frac{1}{7}$ after $\{s_1, \hat{s}_1\}$, to $\mu_2 = \frac{3}{7}$ after $\{s_1, \hat{s}_2\}$, to $\mu_2 = \frac{4}{7}$ after $\{s_2, \hat{s}_1\}$, and to $\mu_2 = \frac{6}{7}$ after $\{s_2, \hat{s}_2\}$. Also, the *ex ante* probability of sending each of these combinations of signals is $\frac{1}{4}$. At last, let's specify the receiver's strategy. After $\{s_1, \hat{s}_1\}$, she takes a_2; after $\{s_1, \hat{s}_2\}$ and $\{s_2, \hat{s}_1\}$, she takes a_5; after $\{s_2, \hat{s}_2\}$, she takes a_7. Therefore, the actions a_2 and a_7 are induced with probability $\frac{1}{2}$, and sender 1's expected payoff is 2.

This strategy profile forms an SPE path. As has been discussed before, the receiver is taking the best responses. For sender 2, as Figure 4 shows, the lower bound for an equilibrium payoff after the first period is 3, conditional on that s_1 updates his belief to $\frac{2}{7}$ or s_2 to $\frac{5}{7}$. Under this strategy profile, sender 2's continuation payoff is 3, reaching the lower bound. For sender 1, his payoff is generically 0 unless the final belief $\mu_2 \in [\frac{1}{7}, \frac{2}{7}] \cup [\frac{5}{7}, \frac{6}{7}]$. One can check that the lower bound for sender 1's equilibrium payoff is 0. Here his continuation payoff is 2, which means this is an equilibrium payoff for him.

This SPE outcome has a feature that it cannot be achieved in any SPE where only one sender reveals information on the equilibrium path. Suppose only sender 1 reveals information. Then to achieve an expected payoff higher than 2, he must induce beliefs $\mu_1 \in [\frac{1}{7}, \frac{2}{7}] \cup [\frac{5}{7}, \frac{6}{7}]$. Yet it is nonequilibrium behavior for sender 2 to *not* further reveal information and only receives payoff 1 in either case. From Figure 4, it can be seen that 1 is lower than the lower bound 2 at any belief $\mu_1 \in [\frac{1}{7}, \frac{2}{7}] \cup [\frac{5}{7}, \frac{6}{7}]$.

On the other hand, if only sender 2 reveals information, then after period 1, $\mu_1 = \mu_0 = \frac{1}{2}$. From Figure 4, the lower bound for sender 2 is 4. One typical signal structure that sender 2 can use in equilibrium with the highest probability of inducing posteriors $\mu_2 \in [\frac{1}{7}, \frac{2}{7}] \cup [\frac{5}{7}, \frac{6}{7}]$ can be described as below:

$$\begin{aligned} \pi_2(\hat{s}_1|\omega_1) &= \frac{1}{14} & \pi_2(\hat{s}_2|\omega_1) &= \frac{5}{28} & \pi_2(\hat{s}_3|\omega_1) &= \frac{3}{4} \\ \pi_2(\hat{s}_1|\omega_2) &= \frac{5}{28} & \pi_2(\hat{s}_2|\omega_2) &= \frac{1}{14} & \pi_2(\hat{s}_3|\omega_2) &= \frac{3}{4} \end{aligned}$$

So after $\hat{s}_1$, the posterior belief is $\mu_2 = \frac{2}{7}$, after $\hat{s}_2$ $\mu_2 = \frac{5}{7}$, and after $\hat{s}_3$ $\mu_2 = \frac{1}{2}$. Let the receiver take a_2, a_7, and a_5 after $\hat{s}_1$, $\hat{s}_2$, and $\hat{s}_3$, respectively. In this way, sender 2's expected payoff is 4, reaching the lower bound for equilibrium. Also, there is probability $\frac{1}{4}$ that either $\hat{s}_1$ or $\hat{s}_2$ will be sent, so sender 1's expected payoff is 1. This is the highest equilibrium payoff for sender 1 if he does not reveal any information.

Therefore, if only one sender reveals information, sender 1 cannot receive an equilibrium payoff as high as 1.

D.2 Proof of Theorem 3

Proposition 7. *Given that the receiver breaks ties in favor of later senders and senders remain silent facing beliefs in the silent sets, the following results hold. For any $t = 1, \ldots, T-1$,*

1. *V_{t+1}^k is continuous in $\mathcal{S}^{t+1}$ for $k > t$.*
2. *$\mathcal{S}^{t+1}$ is closed.*
3. *V_t^t is USC in $\mathcal{S}^{t+1}$.*
4. *There is a solution to (P), σ_t^M, which is uninformative for any $\mu_{t-1} \in \mathcal{S}^t$.*

$$\begin{aligned} \sigma_t^M(\mu_{t-1}) \in \arg\max_{\pi_t \in \Pi} & \sum_{\mu_t \in \text{supp}(\tau_t)} \tau_t(\mu_t|\pi_t, \mu_{t-1}) \cdot V_t^t(\mu_t) \\ s.t. \quad & \text{supp}(\tau_t) \subseteq \mathcal{S}^{t+1} \end{aligned} \tag{P}$$

5. *$\text{supp}(\tau_t) \subseteq \mathcal{S}^t$.*
6. *$\mathcal{S}^t \subseteq \mathcal{S}^{t+1}$.*

Proof. For $t = T$,

(1) V_T^T is continuous on $\mathcal{S}^T$.

V_T^T coincides with $\text{cl}(V_T^T)$ on $\mathcal{S}^T$, hence, V_T^T is continuous on $\mathcal{S}^T$.

(2) $\mathcal{S}^T$ is closed.

Because V_T^T is continuous on $\mathcal{S}^T$, for any sequence $\{\mu^n\} \subseteq \mathcal{S}^T$, s.t. $\mu^n \to \mu$. $\exists$ a subsequence $\{n_k\}$ s.t. $\rho^M(\mu^{n_k}) \to \alpha' \in \arg\max_{\alpha \in \Delta(A)} \mathbb{E}[u(\alpha,\omega)|\mu]$. Because the receiver favors sender T, $V_T^T(\mu) \geq \mathbb{E}[v^T(\alpha',\omega)|\mu] = \lim_{k\to\infty} \mathbb{E}[v^T(\rho(\mu^{n_k}),\omega)|\mu^{n_k}] \geq \text{cl}(V_T^T)(\mu)$. Therefore, $V_T^T(\mu) = \text{cl}(V_T^T)(\mu)$ and $\mu \in \mathcal{S}^T$.

(3) V_{T-1}^{T-1} is USC in $\mathcal{S}^T$.

By the definition of $\mathcal{S}^T$, it is optimal for sender T to send null signals when $\mu_{T-1} \in \mathcal{S}^T$, which is prescribed in the construction of σ_T^M. So $V_{T-1}^{T-1} = V_T^{T-1}$ on $\mathcal{S}^T$. For any sequence of beliefs $\{\mu^n\} \subseteq \mathcal{S}^T$ s.t. $\mu^n \to \mu$, $\exists$ a subsequence $\{\mu^{n_k}\}$ that converge to the limit superior, $\lim_{k\to\infty} V_{T-1}^{T-1}(\mu^{n_k}) = \limsup_{n\to\infty} V_{T-1}^{T-1}(\mu^n)$ and that $\rho^M(\mu^{n_k}) \to \alpha' \in \arg\max_{\alpha\in\Delta(A)} \mathbb{E}[u(\alpha,\omega)|\mu]$ by the maximum theorem. Because V_T^T is continuous on $\mathcal{S}^T$, α' is an option sender T prefers. Now that sender $T-1$ has the second highest priority to be favored by the receiver,

$$V_{T-1}^{T-1}(\mu) \geq \mathbb{E}[v^{T-1}(\alpha',\omega)|\mu] = \lim_{k\to\infty} V_{T-1}^{T-1}(\mu^{n_k}) = \limsup_{n\to\infty} V_{T-1}^{T-1}(\mu^n)$$

(4) There is a solution to (P), σ_T^M, which is uninformative for any $\mu_{T-1} \in \mathcal{S}^T$.

There is a single sender solution when the receiver favors him in indifference. When $\mu_{T-1} \in \mathcal{S}^T$, $V_T^T(\mu_{T-1}) = \text{cl}(V_T^T(\mu_{T-1}))$, therefore, it is optimal for sender T to send null signals.

(5) $\text{supp}(\tau_T) \subseteq \mathcal{S}^T$

If there is any $\mu^* \in \text{supp}(\tau_T)$ such that $\mu^* \notin \mathcal{S}^T$, it means that $V_T^T(\mu^*) < \text{cl}(V_T^T)(\mu^*)$. By the definition of concave closure, there exists an information policy τ' such that $\mathbb{E}[\tau'] = \mu^*$ and $\sum_{\mu\in\text{supp}(\tau')} \tau'(\mu) \cdot V_T^T(\mu) > V_T^T(\mu^*)$.

Construct another information policy τ'' with $\text{supp}(\tau'') = \text{supp}(\tau_T) \cup \text{supp}(\tau') \backslash \mu^*$. For any $\mu \in \text{supp}(\tau_T)$, let $\tau''(\mu) = \tau_T(\mu)$; for any $\mu \in \text{supp}(\tau')$, let $\tau''(\mu) =$

$\tau_T(\mu^*)\tau'(\mu)$. Then,

$$\begin{aligned}
&\sum_{\mu\in\text{supp}(\tau'')} \tau''(\mu)\cdot V_T^T(\mu) \\
&= \sum_{\mu\in\text{supp}(\tau)\backslash\{\mu^*\}} \tau(\mu)\cdot V_T^T(\mu) + \tau(\mu^*) \sum_{\mu\in\text{supp}(\tau')} \tau'(\mu)\cdot V_T^T(\mu) \\
&> \sum_{\mu\in\text{supp}(\tau)} \tau(\mu)\cdot V_T^T(\mu)
\end{aligned}$$

So it is profitable to deviate to τ''.

Suppose these results hold for $t > i$, then I will show they also hold for $t = i$.

(1') V_{i+1}^k is continuous on $\mathcal{S}^{i+1}$, for $k \geq i+1$.

By induction, V_{i+2}^k is continuous on $\mathcal{S}^{i+2}$, for $k \geq i+2$. Plus $\mathcal{S}^{i+1} \subseteq \mathcal{S}^{i+2}$, so V_{i+1}^k is equal to V_{i+2}^k on $\mathcal{S}^{i+1}$ because of the silence of sender $(i+1)$. That means V_{i+1}^k is continuous on $\mathcal{S}^{i+1}$ for $k \geq i+2$. By definition V_{i+1}^{i+1} coincides with $\text{cl}(V_{i+1}^{i+1})$ on $\mathcal{S}^{i+1}$, hence V_{i+1}^{i+1} is continuous on $\mathcal{S}^{i+1}$.

(2') $\mathcal{S}^{i+1}$ is closed.

In the *silent* set $\mathcal{S}^{i+1}$, $V_{i+1}^k = V_T^k$, $\forall k$. By (1'), V_T^k in continuous on $\mathcal{S}^{i+1}$, for $k \geq i+1$. For any sequence $\{\mu^n\} \subseteq \mathcal{S}^{i+1}$, s.t. $\mu^n \to \mu$, by the maximum theorem there exists a subsequence $\{n_k\}$ s.t. $\rho^M(\mu^{n_k}) \to \alpha' \in \arg\max_{\alpha\in\Delta(A)}$ $\mathbb{E}[u(\alpha,\omega)|\mu]$ and $\mathbb{E}[v^k(\alpha',\omega)|\mu] = V_T^k(\mu)$, for $k \geq i+1$. Because sender $(i+1)$ has the priority after subsequent senders,

$$V_{i+1}^{i+1}(\mu) \geq \mathbb{E}[v^{i+1}(\alpha',\omega)|\mu] = \lim_{k\to\infty} \mathbb{E}[v^{i+1}(\rho^M(\mu^{n_k}),\omega)|\mu^{n_k}] \geq \text{cl}(V_{i+1}^{i+1})(\mu)$$

Therefore, $\mu \in \mathcal{S}^{i+1}$.

(3') V_i^i is USC on $\mathcal{S}^{i+1}$.

From the induction, for any $\mu^i \in \mathcal{S}^i$, following senders stay silent, so $V_i^k = V_T^k$, $\forall k$. Plus, V_T^k is continuous on $\mathcal{S}^{i+1}$, for $k \geq i+1$. For any sequence $\{\mu^n\} \subseteq \mathcal{S}^{i+1}$, s.t. $\mu^n \to \mu$, by the maximum theorem there exists a subsequence $\{n_k\}$ s.t. $\rho^M(\mu^{n_k}) \to \alpha' \in \arg\max_{\alpha\in\Delta(A)} \mathbb{E}[u(\alpha,\omega)|\mu]$ and $\mathbb{E}[v^k(\alpha',\omega)|\mu] = V_T^k(\mu)$, for $k \geq i+1$. Because sender i has the priority after subsequent senders,

$$V_i^i(\mu) \geq \mathbb{E}[v^i(\alpha',\omega)|\mu] = \lim_{k\to\infty} V_i^i(\mu^{n_k}) = \limsup_{n\to\infty} V_i^i(\mu^n)$$

(4') There is a solution to (P), σ_i^M, which is uninformative for any $\mu_{i-1} \in \mathcal{S}^i$.

If $\mu_i \in \mathcal{S}^{i+1}$, by construction any sender k ($k > i$) would stay silent and $\mu_T = \mu_i \in \mathcal{S}^{i+1}$. If $\mu_i \notin \mathcal{S}^{i+1}$, sender $(i+1)$ would spread posteriors μ_{i+1} on $\mathcal{S}^{i+1}$ by induction, and following senders would stay silent so that $\mu_T \in \mathcal{S}^{i+1}$. That means, taking into account sender $(i+1)$'s response, sender i can not achieve better expected payoff than the convex combination of value points in $\mathcal{S}^{i+1}$, and it suffices to look at τ_i with support contained in $\mathcal{S}^{i+1}$. In addition, $\mathcal{S}^{i+1}$ is closed and V_i^i is USC in $\mathcal{S}^{i+1}$, so there is an optimal strategy σ_i^M for sender i that solves (P'). Furthermore, when $\mu_{i-1} \in \mathcal{S}^i$, $V_i^i(\mu_{i-1}) = \mathrm{cl}(V_i^i(\mu_{i-1}))$, so it is optimal for sender i to send null signals.

(5') $\mathrm{supp}(\tau_i) \subseteq \mathcal{S}^i$

From (4'), we known that $\mathrm{supp}(\tau_i) \subseteq \mathcal{S}^{i+1}$ and σ_i^M is an optimal strategy. Then for any $\mu \in \mathrm{supp}(\tau_i)$, it must be that $V_i^i(\mu) = \mathrm{cl}(V_i^i)(\mu)$. (By similar argument for (5).) By the definition of $\mathcal{S}^i$, $\mu \in \mathcal{S}^i$.

(6') $\mathcal{S}^i \subseteq \mathcal{S}^{i+1}$

By the definition of $\mathcal{S}^i$, this claim is true.

□

Combining results 5 and 6 of Proposition 7, we have $\mathrm{supp}(\tau_1) \subseteq \mathcal{S}^1 \subseteq \mathcal{S}^2 \cdots \subseteq \mathcal{S}^T$, which means that the possible posteriors of the first period lie within the intersection of all *silent* sets. Thus, subsequent senders will unequivocally stay silent.

D.3 Proof of Theorem 4

Lemma 5. *For each h_{T-1} and $\mu_{T-1} = \bar{\mu}(h_{T-1})$, suppose sender T uses π_T to induces τ_T and the equilibrium outcome is (τ_T, λ^e). Then, for each $\mu_T \in \mathrm{supp}(\tau_T)$, $\mu_T \in \{\mu \in \Delta(\Omega) \,|\, \bar{V}_T^T(\mu) = \mathrm{cl}(\bar{V}_T^T)(\mu)\}$ and $\mathbb{E}[v^T(\lambda^e(\mu_T), \omega) \,|\, \mu_T] = \mathrm{cl}(\bar{V}_T^T)(\mu_T)$.*

Proof. Suppose conditional on h_{T-1}, sender T uses π_T that sends a signal s with positive probability such that $\bar{\mu}(s) = \mu_T$ and $\mathbb{E}[v^T(\lambda^e(\mu_T), \omega) \,|\, \mu_T] = \bar{v}^T(h_{T-1}, \pi_T, s) < \mathrm{cl}(\bar{V}_T^T)(\mu_T)$. Because it is an SPE,

$$\sum_{\mu \in \mathrm{supp}(\tau_T)} \tau_T(\mu) \cdot \mathbb{E}[v^T(\lambda^e(\mu), \omega) \,|\, \mu] \geq \mathrm{cl}(\underline{V}_T^T)(\mu_{T-1}) = \mathrm{cl}(\bar{V}_T^T)(\mu_{T-1})$$

Define a function $f : \text{supp}(\tau^e) \to \mathbb{R}$ such that $f(\mu) = \mathbb{E}[v^T(\lambda^e(\mu), \omega) \,|\, \mu]$ for $\mu \in \text{supp}(\tau^e) \backslash \{\mu_T\}$ and $f(\mu_T) = \text{cl}(\bar{V}_T^T)(\mu_T)$. Then,

$$\sum_{\mu \in \text{supp}(\tau_T)} \tau_T(\mu) \cdot f(\mu) > \sum_{\mu \in \text{supp}(\tau_T)} \tau_T(\mu) \cdot \mathbb{E}[v^T(\lambda^e(\mu), \omega) \,|\, \mu] = \text{cl}(\bar{V}_T^T)(\mu_{T-1})$$

But since for each $\mu \in \text{supp}(\tau^e)$, $f(\mu) \leq \bar{V}_T^T(\mu)$ and τ_T is a mean-preserving spread of μ_{T-1},

$$\sum_{\mu \in \text{supp}(\tau_T)} \tau_T(\mu) \cdot \lambda(\mu) \leq \text{cl}(\bar{V}_T^T)(\mu_{T-1})$$

Contradiction. That means for each $\mu \in \text{supp}(\tau_T)$, $\mathbb{E}[v^T(\lambda^e(\mu), \omega) \,|\, \mu] = \text{cl}(\bar{V}_T^T)(\mu)$. Because $\mathbb{E}[v^T(\lambda^e(\mu), \omega) \,|\, \mu] \in V_T^T(\mu)$ and $\mathbb{E}[v^T(\lambda^e(\mu), \omega) \,|\, \mu] \leq \bar{V}_T^T(\mu) \leq \text{cl}(\bar{V}_T^T)(\mu)$. It follows that $\mu \in \{\mu' \in \Delta(\Omega) \,|\, \bar{V}_T^T(\mu') = \text{cl}(\bar{V}_T^T)(\mu')\}$. □

For each $t \in T$, define the *silent* set as $\mathcal{S}^t = \{\mu \in \Delta(\Omega) \,|\, \bar{V}_t^t(\mu) = \text{cl}(\bar{V}_t^t)(\mu)\}$. Those are the beliefs at which sender t might choose not to reveal information on the equilibrium path.

It is immediate that any SPE in $\Gamma_{T-1}(h_{T-1})$ is a silent equilibrium, given that there is only sender T in the continuation game. Next, I move on to show that any SPE in $\Gamma_{T-2}(h_{T-2})$ has an equivalent outcome to a silent equilibrium.

Suppose for a history after period $T-2$, $h_{T-2} \in H_{T-2}$, an SPE path in $G(h_{T-2})$ is $(\pi_{T-1}, \{\pi_T(s)\}_{s \in S_{T-1}}, \{\alpha(s)\}_{s \in S_{T-1} \times S_T})$. Let the equilibrium outcome of this SPE be (τ^e, λ^e).

Replace π_{T-1} with $\hat{\pi}_{T-1}$ that induces τ^e, let sender T use uninformative signal structures following any signal from $\hat{\pi}_{T-1}$, and let the receiver play the same strategy $\hat{\alpha}$ conditional on the realization of the same posterior belief from the original equilibrium path, i.e., if $s \in S_{T-1}$ induces the same posterior belief as $s' \in S_{T-1} \times S_T$, then $\hat{\alpha}(s) = \alpha(s')$. Since sender $T-1$ does not reveal information on this path, when denoting this path I neglect π_T; that is, the path is $(\hat{\pi}_{T-1}, \{\hat{\alpha}(s)\}_{s \in S_{T-1}})$. Furthermore, the outcome on this path is exactly the same as that under the original path, (τ^e, λ^e).

Then I will show that this is an equilibrium path in $\bar{\Gamma}(h_{T-2})$. For the receiver, after each $s \in S_{T-1}$, she updates her belief to the posterior of some $s' \in S_{T-1} \times S_T$, and she takes the same action as if she received s' under the original strategy profile. Since the original strategy is an equilibrium strategy, here she also maximizes her expected payoff.

By Lemma 5, for each $\mu \in \mathrm{supp}(\tau^e)$, $\mu \in \mathcal{S}^T$ and $\mathbb{E}[v^T(\lambda^e(\mu),\omega) \,|\, \mu] = \mathrm{cl}(\bar{V}_T^T)(\mu)$. So after period $T-1$, no matter which posterior $\mu \in \mathrm{supp}(\tau^e)$ realizes, sender T does not have an incentive to deviate to other signal structure. For sender $T-1$, on these two paths his expected payoffs are the same, which should be higher than $\mathrm{cl}(\underline{V}_{T-1}^{T-1})(\bar{\mu}(h_{T-2}))$ since the original path is an SPE path. By Theorem 1, $\hat{\pi}_{T-1}$ is the signal structure on the equilibrium path.

In summary, I have shown that any SPE outcome in $\bar{\Gamma}(h_{T-2})$ is supported by a silent equilibrium. Notice that the set of continuation payoffs V_{T-2}^k is invariant even if we only focus on silent equilibria in $\bar{\Gamma}(h_{T-2})$.

Induction, $1 \le t < T-2$

Let $\bar{\Gamma}^*(h_t)$ be the set of SPE paths where only sender $t+1$ reveals information on the equilibrium path. Suppose for some $t \in T$, any SPE outcome in $\bar{\Gamma}(h_t)$, for any $h_t \in H_t$, is supported on a silent equilibrium path in $\bar{\Gamma}^*(h_t)$. By similar arguments to Lemma 5, I have the following lemma.

Lemma 6. *For each $h_t \in H_t$ and $\bar{\gamma}_t \in \bar{\Gamma}^*(h_t)$, suppose under $\bar{\gamma}_t$, sender $t+1$ uses π_{t+1} to induce τ_{t+1} and the equilibrium outcome is (τ^e, λ^e). Then, for each $\mu \in \mathrm{supp}(\tau_{t+1})$, $\mu \in \mathcal{S}^{t+1}$ and $\mathbb{E}[v^{t+1}(\lambda^e(\mu),\omega) \,|\, \mu] = \mathrm{cl}(\bar{V}_{t+1}^{t+1})(\mu)$.*

Proof. It can be viewed as a single-sender problem and its proof is similar to that of Lemma 5. □

Then I prove that any equilibrium outcome from $\bar{\Gamma}(h_{t-1})$ is equivalent to an equilibrium outcome from $\bar{\Gamma}^*(h_{t-1})$.

For each $\bar{\gamma}_{t-1} \in \bar{\Gamma}(h_{t-1})$, under $\bar{\gamma}_{t-1}$, π_t is sender t's signal structure. By induction, $\bar{\gamma}_{t-1}$ has an equivalent outcome to another SPE path $\bar{\gamma}_{t-1}^* = (\pi_t, \{\bar{\gamma}_t^*(s)\}_{s \in S_t}, \{\alpha^*(s)\}_{s \in S_t \times S_{t+1}})\}$, where $\bar{\gamma}_t^*(s) \in \bar{\Gamma}^*(h_{t-1}, \pi_t, s)$.

Now let (τ^e, λ^e) be the equilibrium outcome of $\bar{\gamma}_{t-1}$ and $\bar{\gamma}_{t-1}^*$. Think of another path where only sender t reveals information, $\bar{\gamma}'_{t-1} = (\pi'_t, \{\bar{\gamma}'_t(s)\}_{s \in S_t}, \{\alpha'(s)\}_{s \in S_t})$, that has the following features. First, π'_t induces τ^e. Second, for each $s \in S_t$, all subsequent senders $t+1, \ldots, T$ do not reveal information on $\bar{\gamma}'_t(s)$. Third, $\alpha'(s')$ induces the same action as $\alpha^*(s^*)$ when s' and s^* induce the same posterior on $\bar{\gamma}'_{t-1}$ and $\bar{\gamma}_{t-1}^*$, respectively.

Next, I prove that $\bar{\gamma}'_{t-1} \in \bar{\Gamma}(h_{t-1})$.

Senders $t+2, \ldots, T$ optimize because $\bar{\gamma}_t^*(s) \in \bar{\Gamma}_t^*(h_{t-1}, \pi_t, s)$, for each $s \in S_t$. Under $\bar{\gamma}_t^*(s)$, each sender $t+2, \ldots, T$ updates his belief to some $\mu \in \mathrm{supp}(\tau^e)$ and they all keep silent. Under $\bar{\gamma}'_{t-1}$, after period $t+1$, the belief is updated to some $\mu \in \text{supp}(\tau^e)$, plus the receiver reacts in the same way, so it still forms an SPE path for senders $t+2,\dots,T$ to keep silent.

Then consider the optimization of sender $t+1$. Under $\bar{\gamma}'_{t-1}$, the posterior belief after period t would be $\mu_t \in \text{supp}(\tau^e)$ (induced by signal s_t). By Lemma 6,

$$\bar{v}^{t+1}(h_{t-1},\pi'_t,s_t,\bar{\gamma}'_t(s_t)) = \mathbb{E}[v^{t+1}(\lambda^e(\mu_t),\omega)\,|\,\mu_t] = \text{cl}(\bar{V}^{t+1}_{t+1})(\mu_t)$$

The first equation holds because the subsequent senders after sender $t+1$ all keep silent, inducing no change to the belief μ_t. Even if sender $t+1$ provides no information, his payoff is still equal to $\text{cl}(\bar{V}^{t+1}_{t+1})(\mu_t) \geq \text{cl}(\underline{V}^{t+1}_{t+1})(\mu_t)$.

Because $\bar{\gamma}_{t-1}$ is an SPE path, sender t's payoff deriving from that path is no less than $\text{cl}(\underline{V}^t_t)(\bar{\mu}(h_{t-1}))$. Given that sender t's payoff from $\bar{\gamma}'_{t-1}$ is equal to that from $\bar{\gamma}_{t-1}$ (they have the same outcome), the necessary and sufficient condition for an equilibrium path is satisfied according to Theorem 1.

Finally, I can conclude that $\bar{\gamma}'_{t-1}$ is an SPE path in $\bar{\Gamma}^*(h_{t-1})$ that produces the equivalent outcome to $\bar{\gamma}_{t-1}$.